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THE ANOMALOUS DIFFUSION OF WAVE DISTURBANCES IN HYDRODYNAMIC-TYPE SYSTEMS†

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It is shown that solutions of many model non-linear equations of the hydrodynamic type (including the Navier–Stokes equations in the theory of a viscous fluid) in the form of localized wave disturbances undergo anomalous decay and the diffusion-type spreading under the action of Gaussian additive force noise and multiplicative parametric noise that depend randomly on time. The universality of this anomaly (i.e. its independence of the specific form of the equations and, in many respects, of the characteristic properties of the noise) is demonstrated; this reflects the generalized Galilean invariance of the hydrodynamic-type equations and the Gaussian property of random sources. The integrability of certain important hydrodynamic models enables the general conclusions to be supported by particular analytic results. Examples of expressions for the mean flow velocities, developed under the action of random forces on algebraic solitons of the completely integrable equations: equations for internal waves (the Benjamin–Ono equations), the two-dimensional Korteweg–de Vries equation (the Petviashvili–Kadomtseve equation), and the two-dimensional non-linear Schrödinger equation (the Davey–Stewartson equations), are presented in terms of the error function. It is established that for stochastic equations of the hydrodynamic type rapidly decaying multiplicative noise leads to normal Fick diffusion only when there is no additive force noise. Force noise produces anomalous diffusion of wave disturbances, which is described by Richardson's law (with a cubic relation between the temporal and squared spatial scales). The result obtained confirms the universal nature of Richardson's law, which has been demonstrated previously for many examples of turbulent and wave stochastic processes [1, 2]. © 2003 Elsevier Ltd. All rights reserved.

1. THE DIFFUSION OF PARTICLES AND LOCALIZED WAVE DISTURBANCES

The diffusion of a substance has the classic form of Fick diffusion only in sufficiently homogeneous media with a simple structure and small deviations from a state of thermodynamic equilibrium. For this process the quadratic relation between the temporal and spatial scales

$$t \sim \lambda^2$$

is characteristic. In media with a complicated structure, disordered systems, and processes with thermodynamically strong non-equilibrium turbulent variations of the state, the scale relation changes its asymptotic form

$$t^\gamma \sim \lambda^2, \quad \gamma \neq 1, \quad t \to \infty$$

The diffusion acquires an "anomalous" feature. A reduction in the rate of diffusion ("subdiffusion" which $\gamma < 1$) and an increase ("superdiffusion" with $\gamma > 1$) are both possible. In certain cases the asymptotic power-law relations are changed into logarithmic ones. Anomalous diffusion has been observed in many media with a disordered structure (in amorphous solids, polymer melts and solutions, porous media, ionic superconductors, biosystems, etc.), in turbulent flows of fluids, for Brownian particles in shear flows, and under chaotic regimes in dynamical systems [3].

The power-law behaviour, which occurs if there is no a distinctive scale, arises in media with a fractal structure when the structure complication is repeated on changing from one scale to the next. In the case of a fractal structure of the medium the rate of diffusion decreases owing to the repetitive obstacles to the wandering of the particles (such as constrictions, blind alleys, bendings, etc.) in various scales. The power similarity is a specific feature of the turbulent flows of fluids with a cascade of interactions of motions with various scales. Turbulent diffusion provides an example of superdiffusion with the well-known Richardson's law $t^3 \sim \lambda^2$, due to mixing accompanied by energy transfer from one scale to another.

Together with the increased attention in the literature to anomalous diffusion of a substance, the problem of the action of noise on travelling non-linear waves, especially in relation to the completely integrable equations that allow of a complete analytical treatment, has generated considerable interest.

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On the basis of the method of the inverse scattering problem, the results for a number of stochastic equations with the force and parametric noise have been formulated and the diffusion nature of the variations of the mean values and higher-order statistical moments have been established [4, 9]. Within the framework of these investigations the diffusion behaviour was related to the integrability of the equations.

In what follows, using the example of dissipative models of hydrodynamic-type media, to which a viscous fluid belongs, we show that the problem of averaged flow characteristics with time-dependent random forces and parametric velocity noise also leads to the diffusion equation. The model equations treated below, generally speaking, are not completely integrable. It is demonstrated that such a reduction to the diffusion equation and the character of the diffusion anomaly turn out to be closely related to the group structure of the model equations, their generalized Galilean invariance, and the Gaussian character of the external noise. The complete integrability is intriguing only due to the possibility of obtaining specific analytical results. The averaged characteristics, regardless of the specific features of the original differential equations (the universality), turn out to obey the diffusion equation with varying diffusivity. The effect of coloured noise with long-range and short-range correlations on the character of the anomaly is analysed. We shall restrict our consideration by time-dependent Gaussian random sources.

Here only stochasticity produced by random external fluxes and forces is discussed. Stochasticity that originates due to randomness of the initial conditions and the development of a multi-step cascade of hydrodynamic instability is not considered. In the case of a viscous fluid this means that our analysis is restricted to non-turbulent flow regimes.

2. THE HYDRODYNAMIC-TYPE EQUATIONS WITH SOURCES

If pulsating external forces and homogeneous flows are taken into account, fairly general Galilean invariant equations of motion of a continuum medium can be written down in the form of model integro-differential equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{U}(t) \cdot \nabla \mathbf{u} = -\nabla p + \int K(\mathbf{x}-\mathbf{y}) \Delta \mathbf{u} d\mathbf{y} + \mathbf{f} \tag{2.1}$$

For a δ-like form of the integral kernel these equations can be reduced to the Navier–Stokes differential equations for an incompressible viscous fluid (the constant density is put equal to unit). For the same kernel and a constant pressure the Burgers vector model [10] is obtained

$$\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{U}(t) \cdot \nabla \mathbf{u} = \eta \Delta \mathbf{u} + \mathbf{f}$$

For potential flows this model reduces to the Kardar–Parisi–Zhang equation for the potential [11] and to the equation of heat conduction for a logarithmic potential.

In the one-dimensional version, Eq. (2.1) encompasses many cases frequently discussed. If the pressure is constant, the general one-dimensional integro-differential form of the equation corresponds to the Whiteham model [12]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + U(t)\frac{\partial u}{\partial x} = \int K(x-y)\frac{\partial^2 u}{\partial y^2} dy + f$$

from which, when the integral kernel has the form of the sum of a δ-function and a finite number of its derivatives, we obtain the physically interesting model differential equations [13, 14]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + U(t)\frac{\partial u}{\partial x} = \sum_{k=0}^{n} \eta_k \frac{\partial^{k+2} u}{\partial y^{k+1}} + f$$

In particular, if we confine ourselves by the sum of a δ-function and its second derivative, we obtain the Kuramoto–Sivashinsky equation [15]. When the kernel has the form of a δ-function and its first derivative we obtain the Korteweg–de Vries–Burgers equation [16], and if the kernel has the form of a derivative of the δ-function, we obtain the Korteweg–de Vries equation [17, 18]. Finally, for kernels $K(x) \sim P[1/x]$ and $K(x) \sim P[\mathrm{ctg}\pi x/(2h) - \mathrm{sgn} x]$ we obtain completely integrable integro-differential equations: the Benjamin-Ono equation and the equation for internal waves in a basin of finite depth [18].

If the force is time-dependent, using the change of variables

$$\mathbf{x} = \boldsymbol{\xi} + \mathbf{x}_0(t), \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\boldsymbol{\xi}, t) + \mathbf{u}_0(t), \quad \frac{d\mathbf{u}_0}{dt} = \mathbf{f}(t), \quad \frac{d\mathbf{x}_0}{dt} = \mathbf{u}_0(t) + \mathbf{U}(t) \tag{2.2}$$

which corresponds to transferring to an accelerating frame of reference, one can eliminate the external flux and force from the equations. Then

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v} = -\nabla p + \int K(\mathbf{x}-\mathbf{y})\Delta\mathbf{v}d\mathbf{y} \tag{2.3}$$

This establishes a relation between the solutions of the stochastic equations (2.1) with a random force and uniform flux (parametric noise) and the deterministic equation (2.3). The change of variables (2.2) can be written in the form

$$\begin{gathered} \mathbf{u}(\mathbf{x}, t) - \mathbf{u}_0(t) = \exp(-\mathbf{x}_0(t)\nabla)\mathbf{v}(\mathbf{x}, t) \\ \mathbf{x}_0(t) = \int_0^t (\mathbf{U}(\tau) + (t-\tau)\mathbf{f}(\tau))d\tau, \quad \mathbf{u}_0(t) = \int_0^t \mathbf{f}(\tau)d\tau \end{gathered} \tag{2.4}$$

by using the operator of stochastic shift and putting $\mathbf{x}_0(0) = \mathbf{u}_0(0) = 0$. The deviation of the velocities from the "background" pulsations $\mathbf{u}_0(t)$ is represented in the form of the product of random and deterministic factors that is convenient for obtaining relations for the averaged characteristics.

3. A COMPRESSIBLE FLUID, SHALLOW WATER, AND THE NON-LINEAR SCHRÖDINGER EQUATION

More general Galilean invariant models of compressible media are treated similarly. Thus, for a compressible isoentropic fluid the stochastic equation of motion, together with the mass conservation equation and the consequence of isoentropicity (w is the enthalpy)

$$\begin{gathered} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} + \mathbf{U}(t))\nabla\mathbf{u} = -\nabla w + \int K(\mathbf{x}-\mathbf{y})\Delta\mathbf{u}d\mathbf{y} + \mathbf{f}(t) \\ \frac{\partial \rho}{\partial t} + (\mathbf{u} + \mathbf{U}(t))\nabla\rho + \rho(\nabla\mathbf{u}) = 0, \quad w = w(\rho) \end{gathered} \tag{3.1}$$

by using the change of variables, generalizing replacement (2.2)

$$\begin{gathered} \mathbf{x} = \boldsymbol{\xi} + \mathbf{x}_0(t), \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\boldsymbol{\xi}, t) + \mathbf{u}_0(t), \quad \rho(\mathbf{x}, t) = \mathrm{P}(\boldsymbol{\xi}, t) \\ \frac{d\mathbf{u}_0}{dt} = \mathbf{f}(t), \quad \frac{d\mathbf{x}_0}{dt} = \mathbf{u}_0(t) + \mathbf{U}(t) \end{gathered}$$

is reduced to equations without pulsating sources, that is, the solution of the stochastic equations is reduced to solving deterministic equations.

As is well known, for a compressible ideal gas with a constant ratio of the heat capacities equal to two, when $w(\rho) = -\kappa\rho$, and if there is no dissipation and dispersion ($K = 0$), the gas dynamic equations reduce to the classic shallow-water equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} + \mathbf{U}(t)) \cdot \nabla\mathbf{u} = \kappa\nabla h + \mathbf{f}(t), \quad \frac{\partial h}{\partial t} + (\mathbf{u} + \mathbf{U}(t)) \cdot \nabla h + h\nabla \cdot \mathbf{u} = 0 \tag{3.2}$$

The role of the density is transferred to the water depth ($\rho \to h$), and the pulsating sources $\mathbf{f}(t)$ and $\mathbf{U}(t)$ can be removed by the change of variables introduced above.

Other physically important equations – the non-linear Schrödinger equations, are related to the shallow-water equations. It can be verified by direct substitution that, using the amplitude-phase change of the variables, known as the Madelung transformation (originally this transformation was used by him to write the equations of quantum mechanic in the form of hydrodynamic equations [19, 20])

$$\psi = ae^{i\varphi}, \quad a = |\psi| = \sqrt{h}, \quad \mathbf{u} = \nabla\varphi, \quad \varphi = -i\ln(\psi/|\psi|)$$

the non-linear Schrödinger equations for the complex ψ-function

$$i\frac{\partial\psi}{\partial t}+i\mathbf{U}(t)\cdot\nabla\psi+\frac{1}{2}\nabla^2\psi+\mathbf{x}\mathbf{f}(t)\psi+\kappa|\psi|^2\psi-\frac{\nabla^2|\psi|}{2|\psi|}\psi = 0 \quad (3.3)$$

are transformed into the classic shallow-water equations. If the amplitude variations are much smaller than the phase variations, the last term in the equation can be ignored, and, if there are no random sources, we arrive at the famous non-linear Schrödinger equation with cubic non-linearity, which is completely integrable in the one-dimensional case.

The equations for the ψ-function with considerable non-linearity and of higher order correspond to the shallow-water equations with dissipation and dispersion. For example, by applying the Madelung transformation to the integrable "classical Boussinesq system".

$$\frac{\partial u}{\partial t}+uu_x+h_x = 0, \quad \frac{\partial h}{\partial t}+(uh)_x = \mu u_{xxx}$$

it can be made to correspond to an equation of the form

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\psi_{xx}-|\psi|^2\psi-\frac{|\psi|_{xx}}{2|\psi|}\psi-\frac{\mu}{2\psi}\left(\ln\frac{\psi}{|\psi|}\right)_{xxxx} = 0$$

4. THE DIFFUSION SMOOTHING OF DISTURBANCES

If we restrict our analysis to statistically isotropic Gaussian sources, the characteristics $\mathbf{x}_0(t)$ and $\mathbf{u}_0(t)$, which depend on them linearly, will have the same properties. Then averaging of the random shift operator in relation (2.4) gives

$$\langle\mathbf{u}(\mathbf{x},t)\rangle = \exp\left(\frac{\langle\mathbf{x}_0^2(t)\rangle}{2d}\Delta\right)\mathbf{v}(\mathbf{x},t) \quad (4.1)$$

and if the consequences of the isotropy and the Gaussian properties of the random processes $\mathbf{U}(t)$ and $\mathbf{f}(t)$

$$\langle\mathbf{U}\rangle = \langle\mathbf{f}\rangle = 0, \quad \langle U_iU_j\rangle = d^{-1}\langle\mathbf{U}^2\rangle\delta_{ij}, \quad (f_if_j) = d^{-1}\langle\mathbf{f}^2\rangle\delta_{ij}, \quad \langle\mathbf{x}_0\rangle = 0$$

$$\langle\exp(-\mathbf{x}_0(t)\cdot\nabla)\rangle = \exp\left(\frac{\langle\mathbf{x}_0^2(t)\rangle}{2d}\Delta\right)$$

are taken into account. Here d is the dimensionality of the problem.

By differentiating expression (4.1) with respect to time it can be shown that, when the initial distribution is stationary (before the source was switched on), the mean velocity satisfies the diffusion equation with variable coefficient

$$\frac{\partial\langle\mathbf{u}\rangle}{\partial t} = D(t)\Delta\langle\mathbf{u}\rangle, \quad D(t)\equiv\frac{1}{2d}\frac{\partial\langle\mathbf{x}_0^2(t)\rangle}{\partial t} \quad (4.2)$$

Taking into account the statistical connection between the statistically uniform force and velocity sources and using relation (2.4), we obtain

$$\langle\mathbf{x}_0^2(t)\rangle = \int_0^t(t-\tau)\left[2R_U(\tau)+2tC(\tau)+\frac{1}{3}(2t^2-t\tau-\tau^2)R_f(\tau)\right]d\tau$$

$$R_U(\tau)\equiv\langle\mathbf{U}(0)\cdot\mathbf{U}(\tau)\rangle, \quad R_f(\tau)\equiv\langle\mathbf{f}(0)\cdot\mathbf{f}(\tau)\rangle, \quad C(\tau)\equiv\langle\mathbf{U}(0)\cdot\mathbf{f}(\tau)\rangle$$

$$D(t) = \frac{1}{d}\int_0^t R_U(\tau)d\tau+\frac{1}{d}\int_0^t(2t-\tau)C(\tau)d\tau+\frac{t}{d}\int_0^t(t-\tau)R_f(\tau)d\tau$$

For δ-correlated random sources (white noise) we have

$$R_f(t) = a_f\delta(t),\quad R_U(t) = a_U\delta(t),\quad C(t) = \gamma\sqrt{a_U a_f}\delta(t),\quad \gamma \le 1$$
$$\langle \mathbf{x}_0^2(t)\rangle = \frac{1}{3}a_f t^3 + \gamma\sqrt{a_U a_f}t^2 + a_U t \tag{4.3}$$

and for long times the asymptotic behaviour of the mean square value of the spread of the velocity distributions turns out to be normal or abnormal depending on whether there is a force source or not

$$\langle \mathbf{x}_0^2(t)\rangle \sim \begin{cases} t, & a_f = 0 \\ t^3, & a_f \ne 0 \end{cases},\quad t\to\infty \tag{4.4}$$

Accordingly, the asymptotic form of the diffusivity will be

$$D(t) = \frac{a_U}{2d} + \frac{\gamma}{d}\sqrt{a_U a_f}t + \frac{a_f}{2d}t^2 \sim \begin{cases} \text{const}, & a_f = 0 \\ t^2, & a_f \ne 0 \end{cases},\quad t\to\infty \tag{4.5}$$

so that it remains constant only for velocity white noise (diffusion with varying diffusivity is sometimes referred to as "strange diffusion" [21]). In the general case for long times force noise makes the main contribution, whereas for short times parametric noise makes the main contribution. As the time increases the diffusion scale exceeds the initial size of disturbance and then becomes decisive.

As it is obvious from relations (4.4) and (4.5), in the first-order approximation the correlation of the two noises has no effect on the asymptotic form of the dispersion and diffusivity. Hence, for simplicity we will restrict ourselves to discussing mutually uncorrelated velocity and force noises.

In the case of coloured noise with short-range correlations the asymptotic result remains the same as in the case of white noise. For example, in the case of exponentially decaying correlations of the sources

$$R_f(t) = \frac{a_f}{2\theta_f}\exp\left(-\frac{|t|}{\theta_f}\right),\quad R_U(t) = \frac{a_U}{2\theta_U}\exp\left(-\frac{|t|}{\theta_U}\right)$$

after integrating we obtain

$$\langle \mathbf{x}_0^2(t)\rangle = a_U\left[t - \theta_U + \theta_U\exp\left(-\frac{t}{\theta_U}\right)\right] + $$
$$+ a_f\left\{\frac{1}{3}t^3 - \frac{1}{2}\theta_f t^2 - \theta_f^2 t\exp\left(-\frac{t}{\theta_f}\right) + \theta_f^3\left[1 - \exp\left(-\frac{t}{\theta_f}\right)\right]\right\} \tag{4.6}$$

$$D(t) = \frac{a_U}{2d}\left[1 - \exp\left(-\frac{t}{\theta_U}\right)\right] + \frac{a_f}{2d}t\left[1 - \theta_f + \theta_f\exp\left(-\frac{t}{\theta_f}\right)\right] \tag{4.7}$$

At long times from this we obtain exactly the same asymptotic formulae (4.4) and (4.5) with a predominant influence of force noise.

In the case of long-range correlations with a power law decay

$$R_U(t) = b_U|t|^{-\beta},\quad R_f(t) = b_f|t|^{-\gamma},\quad |t| > \theta$$

for the root-mean-square spread and instantaneous diffusivity under the action of one source or other we obtain respectively

$$\langle \mathbf{x}_0^2(t)\rangle = b_U\begin{cases} a_0 t^{2-\beta} + a_1 t + a_2, & \beta \ne 1, 2 \\ b_{\beta 0}t^{2-\beta}\ln(t/\theta) + b_{\beta 1}t + b_{\beta 2}, & \beta = 1, 2 \end{cases}$$

$$D(t) = \frac{b_U}{d}\begin{cases} \kappa_0 t^{1-\beta} + \kappa_1, & \beta \ne 1 \\ \kappa_2\ln(t/\theta) + \kappa_3, & \beta = 1 \end{cases}$$

$$\langle \mathbf{x}_0^2(t)\rangle = b_f\begin{cases} c_0 t^{4-\gamma} + c_1 t^3 + c_2 t^2 + c_3, & \gamma \neq 1, 2, 4 \\ d_{\gamma 0} t^{4-\gamma}\ln(t/\theta) + d_{\gamma 1} t^3 + d_{\gamma 2} t^2 + d_{\gamma 3}, & \gamma = 1, 2, 4 \end{cases}$$

$$D(t) = \frac{b_f}{d}\begin{cases} \chi_0 t^{3-\gamma} + \chi_1 t^2 + \chi_2 t, & \gamma \neq 1, 2 \\ \chi_{\gamma 0} t^{3-\gamma}\ln(t/\theta) + \chi_{\gamma 1} t^2 + \chi_{\gamma 2} t, & \gamma = 1, 2 \end{cases}$$

Here the coefficients are expressed in terms of the scale characteristics θ to some power and the integral moments of the source correlation functions.

When there is no statistical relation between the noises, the contributions from each of them are simply summed. From this it follows that under the action of both noises and for long times diffusion spreading of the localized velocity distributions has an anomalous superdiffusion form with a predominant influence of additive force noise

$$\langle \mathbf{x}_0^2(t)\rangle \sim \begin{cases} t^3, & \gamma > 1 \\ t^3 \ln t, & \gamma = 1 \\ t^{4-\gamma}, & 0 < \gamma < 1 \end{cases}, \quad t \to 1 \tag{4.8}$$

If only velocity noise is present, the diffusion spreading will be normal or anomalous (superdiffusion) depending on the rate of decay of the correlation of this multiplicative noise

$$\langle \mathbf{x}_0^2(t)\rangle \sim \begin{cases} t, & \beta > 1 \\ t \ln t, & \beta = 1 \\ t^{2-\beta}, & 0 < \beta < 1 \end{cases}, \quad t \to \infty \tag{4.9}$$

The last result was obtained before [8] in a special case of the Korteweg–de Vries stochastic equation.

Correlations of power form are satisfactory only for long times. However, it is not difficult to present an example of the acceptable behaviour of the correlation functions for all times and a power-law decay for long times

$$R_U(t) = R_U(0)|1 + t/\theta|^{-\beta}, \quad R_f(t) = R_f(0)|1 + t/\theta|^{-\gamma}$$

The dispersion and the diffusivity are then expressed in terms of power and logarithmic functions. For example, in the case of velocity noise we have

$$\frac{\langle \mathbf{x}_0^2(t)\rangle}{2R_U(0)\theta^2} = \begin{cases} \left(1 + \frac{t}{\theta}\right)\ln\left(1 + \frac{t}{\theta}\right), & \beta = 1 \\ \frac{1}{\beta - 1}\left[\frac{t}{\theta} + \frac{1}{2-\beta}\left(1 - \left(1 + \frac{t}{\theta}\right)^{2-\beta}\right)\right], & \beta \neq 1 \end{cases}$$

The asymptotic result for $t \gg \theta$ agrees with relation (4.9).

The rate of decrease of the mean velocities can be estimated from the change in the width of the distribution and from the first integral of the equations of motion. If the mean force is zero, the averaged Eq. (2.1) allows of the integral of the averaged momentum (more exactly, the integral acquires such a meaning after multiplication by the constant density in the case of an incompressible fluid, whereas in the compressible model (3.1) its constancy is possible only with certain restrictions, for example, in the case of potential flow)

$$\int \langle \mathbf{u}(\mathbf{x}, t)\rangle d\mathbf{x} = \text{const}$$

If this integral is non-zero, for the variation in the amplitude of the localized disturbance we have $\langle \mathbf{u}(\mathbf{x}, t)\rangle \sim (\langle \mathbf{x}_0^2(t)\rangle)^{-d/2}$. Actually, using relation (4.1) written in the form of the Fourier expansion

$$\langle \mathbf{u}(\mathbf{x}, t)\rangle = \frac{1}{(2\pi)^d}\int \exp\left(-k^2\frac{\langle \mathbf{x}_0^2(t)\rangle}{2d}\right) < \mathbf{u}\langle \mathbf{k}, 0\rangle > \exp(i\mathbf{k}\mathbf{x})d\mathbf{k}$$

for long times we get the asymptotic estimate

$$\langle \mathbf{u}(\mathbf{x},t)\rangle \approx \left(\frac{2\pi}{d}\langle \mathbf{x}_0^2(t)\rangle\right)^{-d/2} \exp\left(-\frac{d}{2}\frac{\mathbf{x}^2}{\langle \mathbf{x}_0^2(t)\rangle}\right)\int \langle \mathbf{u}(\mathbf{x},0)\rangle d\mathbf{x}$$

A wave packet with a non-zero limited total momentum takes such a Gaussian form at the final stage of degeneration of the disturbances, regardless of their form at the initial instant. However, the rate of decay of the amplitude of the disturbances will be greater if the dimensionality of the problem is greater. According to (4.8) and (4.9) the decay will be slowest in the regime of one-dimension diffusion produced by parametric noise with fairly rapidly decaying correlations (more than the first degree in time). When the total momentum vanishes, the degeneration of the disturbance is accelerated. The asymptotic form of the time-dependence of the disturbances is then determined by the multiple characteristics of the initial distribution.

Both cases are encountered for solutions of the Kadomtsev–Petviashvili (K-P) equation in the form of solitons. Unlike to the multidimensional model equations of the form (2.1), (3.1), in the case of the K-P equations there is no equal status in the spatial variables due to the relative slowness of the transverse variations. The stochastic K-P equations can be written in a form of the system

$$\partial u/\partial t + uu_x + U(t)u_x + u_{xxx} = p_y + f(t), \quad p_x = \pm u_y \tag{4.10}$$

which, when there is no noise, is called the K-P 1 equation if the sign on the right-hand side of the second equation is positive, and the K-P 2 equation if the sign is negative. Such as hydrodynamic model also possesses the generalized Galilean invariance and its equations can be reduced to a homogeneous form without random sources by applying the previous transformation (2.2). From this it follows that relation (4.1) and the diffusion equation (4.2) remain valid for the mean velocity, and results (4.4)–(4.9) are also true for the mean square width of the velocity distribution and the diffusivity. The force noise will continue to be decisive in the anomalous character of the diffusion.

The K-P 2 equations have stable solutions in the form of one-dimensional solitons with sech^2 shape that are identical with the Korteweg–de Vries solitons. Since for such solitons the integral momentum is non-zero, their height decreases as $\langle x_0^2(t)\rangle^{-1/2}$. The mean square spread $\langle x_0^2(t)\rangle$ follows the asymptotic dependences (4.4), (4.8) and (4.9). In the case of the K-P 1 equations the two-dimensional solitons with a power-like spatial decay (algebraic solitons), the total momentum of which vanishes, turn out to be stable. Finally the more rapid decay of the amplitude $\sim\langle x_0^2(t)\rangle$ will correspond to the asymptotic spread of the averaged soliton at the previous rate $\sim\langle x_0^2(t)\rangle^{-1}$. Actually, if there are no noises, the algebraic soliton that moves along the x axis

$$v(x-c_0t, Y) = \frac{\partial}{\partial x}\frac{12}{x-c_0t+iY} + \text{c.c.}, \quad Y \equiv \sqrt{c_0y^2 + 3/c_0}$$

is an exact solution of the K-P 2 equation. If the noise is switched on, this soliton spreads with time, and the averaged velocity field is expressed in terms of the error integral [9]

$$\langle u(x,y,t)\rangle = \frac{12}{\langle x_0^2(t)\rangle}[1 - \sqrt{\pi}\varsigma\exp(\varsigma^2)\mathrm{erfc}\,\varsigma] + \text{c.c.}, \quad \varsigma \equiv \frac{Y + i(x-c_0t)}{(2\langle x_0^2(t)\rangle)^{1/2}}$$

For a such a solution the integral over x vanishes, and the accelerated decay pointed out above occurs.

The other, also completely integrable equation, namely, the Benjamin–Ono equation

$$\frac{\partial v}{\partial t} + vv_x + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v_{yy}}{y-x}dy = 0$$

admits of a solution in the form of a one-dimensional algebraic soliton

$$v(x-c_0t) = \frac{2}{1/c_0 + i(x-c_0t)} + \text{c.c.}$$

In this case the total momentum is non-zero (it equals 4π), and under the action of both types of noise such a soliton decays more slowly [6]

$$\langle u(x,t)\rangle = \left(\frac{\pi}{2\langle x_0^2(t)\rangle}\right)^{1/2}\exp(\xi^2)\operatorname{erfc}\xi + \text{c.c.},\quad \xi \equiv \frac{1/c_0 + i(x - c_0 t)}{(2\langle x_0^2(t)\rangle)^{1/2}}$$

provided the rate of mean square spreading ($\sim\langle x_0^2(t)\rangle$) is the same.

One more example with a simple analytical result is the soliton solution of the completely integrable Davey–Stewartson (DS) equations. The Davey–Stewartson model is a multidimensional generalization of the non-linear Schrödinger equation in which two spatial variables occur in an asymmetrical way, unlike equations of type (3.3). If there are parametric and force sources, these equations have the form of the system

$$i\frac{\partial\psi}{\partial t} + iU(t)\psi_x + \frac{1}{2}(\varepsilon\psi_{xx} + \psi_{yy}) + [-V + \kappa|\psi|^2]\psi = -xf(t),\quad \varepsilon V_{xx} - V_{yy} = 2\varepsilon\kappa|\psi|^2_{xx}$$

Here $\kappa = \pm 1$ and $\varepsilon = \pm 1$ (if $\varepsilon = +1$, we have the DS I model, and if $\varepsilon = -1$, we have the DS II model). Like the case of the more symmetrical equation, using the change of variables analogous to (2.2)

$$\psi(x,y,t) = \Psi(x - x_0(t), y, t)e^{i\delta(x,y,t)},\quad V(x,y,t) = W(x - x_0(t), y, t)$$

$$x_{0t} = U(t) + \int_0^t f(\tau)d\tau,\quad \delta(x,y,t) = x\int_0^t f(\tau)d\tau + \delta_0(t),\quad 2\delta_{0t} = U^2(t) - x_{0t}^2(t)$$

one can establish a relation between the solutions of the stochastic equations and the solutions of the corresponding deterministic equations with vanishing noise.

The DS II equations ($\varepsilon = -1$) without random sources for $\kappa = -1$ allow of a solution in soliton form with algebraic envelope (in all directions the amplitude decays in a power form)[22]

$$A(X,Y) = \frac{2|\nu|}{X^2 + Y^2},\quad X \equiv x - 4t\operatorname{Im}\lambda,\quad Y \equiv \sqrt{(y + 4t\operatorname{Re}\lambda)^2 + |\nu|^2}$$

If the noise is Gaussian, the result for the averaged soliton intensity, like the preceding case, is expressed in terms of the error function.

$$\langle|\psi|^2\rangle = \sqrt{\frac{\pi}{\tau}}\frac{|\nu|^2}{2Y^3}(1 - Y\partial_Y)\exp(\varsigma^2)\operatorname{erfc}\varsigma + \text{c.c.},\quad \varsigma = \frac{Y + iX}{2\sqrt{\tau}}$$

$$2\tau \equiv \langle x_0^2(t)\rangle = \int_0^t\left[2(t-\tau')R_U(\tau') + (t-\tau')^2\frac{2t+\tau'}{3}R_f(\tau')\right]d\tau'$$

The change in this mean characteristics is described by the diffusion equation

$$\frac{\partial\langle|\psi|^2\rangle}{\partial t} = D(t)\langle|\psi|^2\rangle_{xx},\quad D(t) = \frac{1}{2}\frac{\partial\langle x_0^2(t)\rangle}{\partial t}$$

Since the nature of the diffusion is anomalous, the spread of the signal intensity follows the asymptotic ($t \to \infty$) law $\langle x_0^2(t)\rangle \sim t^3$ for noise or coloured noise with short-range (rapidly decaying) correlations. For parametric noise diffusion of this type turns out to be normal ($\langle x_0^2(t)\rangle \sim t$). For coloured noise with slowly decaying correlations the soliton diffusion is always anomalous.

In the multidimensional cases, anisotropy of external noise sources can lead to anisotropy of the anomalous diffusion, and cases are possible when the directions can be distinguished with respect to super-, sub- or normal diffusion.

Assume that the multiplicative velocity source and the additive force source are statistically anisotropic, homogeneous with zero means, and not cross correlated. Then after averaging relations (2.4), which also hold in this case, we obtain a generalization of the diffusion equation

$$\frac{\partial\langle\mathbf{u}\rangle}{\partial t} = D_{\alpha\beta}(t)\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\langle\mathbf{u}\rangle, \quad D_{ij}(t) \equiv \frac{1}{2}\frac{\partial\langle x_{0i}(t)x_{0j}(t)\rangle}{\partial t}$$

The non-diagonal diffusion tensor reflects the diffusion anisotropy. In the case of uniaxial anisotropy of the noises, when the **n** axis stands out, the correlation tensors of the sources have the form

$$\langle U_i(0)U_j(t)\rangle = R_{II}(t)n_in_j + R_\perp(t)(\delta_{ij} - n_in_j)$$
$$\langle f_i(0)f_j(t)\rangle = F_{II}(t)n_in_j + F_\perp(t)(\delta_{ij} - n_in_j)$$

and, by virtue of the linearity of the relations, for the diffusivities we also have

$$D_{ij}(t) = D_{II}(t)n_in_j + D_\perp(t)(\delta_{ij} - n_in_j)$$

If we confine ourselves to the example of a transverse ($F_{II}(t) = 0$) random force, only the multiplicative velocity noise will affect the longitudinal diffusion

$$D_{II}(t) = \int_0^t R_{II}(\tau)d\tau, \quad D_\perp(t) = \int_0^t R_\perp(\tau)d\tau + t\int_0^t (t-\tau)F_\perp(\tau)d\tau$$

and according to relations (4.3)–(4.9) normal longitudinal diffusion occurs with rapidly decaying noise correlations and anomalous transverse diffusion. If the random force is longitudinal ($F_{II}(t) = 0$), conversely, normal transverse diffusion with longitudinal superdiffusion is possible.

A diffusion form of the variation also occurs for many functions of higher-order moments. As the example, for an arbitrary power of the velocity deviation from the pulsating background value $\mathbf{u}_0(t)$, we obtain

$$|\delta\mathbf{u}|^n \equiv (\mathbf{u}(\mathbf{x}, t) - \mathbf{u}_0(t))^n = \mathbf{v}^n(\mathbf{x} - \mathbf{x}_0(t), t) = \exp(-\mathbf{x}_0(t)\nabla)\mathbf{v}^n(\mathbf{x}, t)$$

and in the case of statistically isotropic sources, after averaging, we obtain a formula for the moments quite similar to (4.1). From that formula there follows the same diffusion equation (4.2) for higher-order moments.

5. CONCLUSION

Thus, within the framework of the hydrodynamic-type models it is possible to recognize the universal nature of the smoothing of the initially localized disturbances under the joint action of a statistically uniform force (also isotropic in the multidimensional case) and parametric noise of Gaussian form. Variations of the mean values and some other moment characteristics are described by a linear diffusion equation irrespective of the specific features of the non-linear equations of the model. This behaviour is connected with the invariance of the models discussed under generalized Galilean transformations and the Gaussian properties of the external random sources. The integrability of the model equations only helps observable analytical results to be obtained with an assurance that solutions exist in the form of a localized travelling wave (soliton) when there is no noise.

Under the joint action of additive and multiplicative noise, that depend only on time, additive force noise of arbitrary spectral form (with arbitrary rate of decay of the correlation) plays a governing role in the anomalous character of the diffusion process. For a rapid decay of the force correlations (δ-like correlations or those decaying more rapidly than the first power in time) the anomaly takes the simple universal form of Richardson's law $\langle \mathbf{x}_0^2(t)\rangle \sim t^3$. The anomaly of the diffusion is enhanced (we then have $\langle \mathbf{x}_0^2(t)\rangle \sim t^3 \ln t$ if $\gamma = 1$ or $\sim t^{4-\gamma}$ if $\gamma < 1$) when the force correlations decay in inverse proportion to the first power in time or more slowly. Under the action of only multiplicative (parametric) noise the diffusion becomes normal ($\langle \mathbf{x}_0^2(t)\rangle \sim t$) for sufficiently rapid decay of the correlations. Hyperbolic or slower decays are also exceptional here.

In multidimensional problems with anisotropic noise the diffusion can be normal in one direction with superdiffusion in a direction orthogonal to it, depending on the type of anisotropy of the force source.

REFERENCES

1. NOVIKOV, Ye. A., The method of random forces in the theory of turbulence. *Zh. Eksp. Teor. Fiz.*, 1864, **44**, 6, 2159–2168.
2. GOLITSYN, G. S., Methodological foundations of the theory of turbulence and of sea waviness. *Izv. Akad. Nauk. Fiz. Atm. Okeana*, 2001, **37**, 4, 438–445.
3. BOUCHAUD, J. P. and GEORGES, A. Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. *Phys. Rep.*, 1990, **195**, 4–5, 127–293.
4. WADATI, M., Stochastic Kortweg–de Vries equation. *J. Phys. Soc. Jap.*, 1983, **52**, 8, 2642–2648.
5. MAIMISTOV, A. I. and MANYKIN, E. A., Propagation of solitons in heterogeneous and random media of special type. *Izv. Vazov. Fizika*, 1987, **30**, 4, 91–97.
6. GORODTSOV, V. A., Diffusion spreading of localized hydrodynamic disturbances under the action of random forces. *Prikl. Mat. Mekh.*, 1988, **52**, 2, 211–217.
7. ABDULLAYEV, F. H., *Dynamic Chaos of Solitons*. Fan, Taskent, 1990.
8. IIZUKA, T., Anomalous diffusion of solitons in random system. *Phys. Lett. Ser. A.*, 1993, **181**, 1, 39–42.
9. GORODTSOV, V. A., The stochastic Kadomtsev–Petviashvili equation. *Zh. Eksp. Teor. Fiz.*, 2000, **90**, 6, 1270–1279.
10. BURGES, J. M., *The Nonlinear Diffusion Equation*. Reidel, Dordrecht, 1974. 179.
11. KARDAR, M., PARISI, G., ZHANG, Y. C., Dynamic scaling of growing interfaces. *Phys. Rev., Lett.*, 1986, **56**, 9, 889–892.
12. WHITHAM, G. B., Variational methods and applications to water waves. *Proc. Roy. Soc. London. Ser. A*, 1967, **299**, 1456, 6–25.
13. NIKOLAYEVSKII, V. N., Viscoelasticity with internal oscillators as a possible model of a seismically active medium. *Dokl. Akad. Nauk SSSR*, 1985, **283**, 6, 1321–1324.
14. NIKOLAEVSKII, V. N., Dynamics of viscoelastic media with internal oscillators. *Lecture Notes in Engineering*. Springer, 1989, 39, 210–221.
15. HYMAN, J. M., NIKOLAENKO, B. and ZALESKI, S., Order and complexity in the Kuramoto–Sivashinsky model of weakly turbulent interfaces. *Physica. Ser. D.*, 1986, **23**, 1–3, 265–292.
16. JOHNSON, R. S., A non-linear equation incorporating damping and dispersion. *J. Fluid Mech.*, 1970, **42**, Part 1, 49–60.
17. ZAKHAROV, V. Ye., MANAKOV, S. V., NOVIKOV, S. P. and PITAYEVSKII, L. P., *The Theory of Solitons: The Method of the Inverse Scattering Problem*. Nauka, Moscow, 1980.
18. ABLOWITZ, M. J. and SEGUR, H., *Solitons and the Inverse Scattering Transform*. SIAM, Philadelphia, 1981.
19. Madelung, e., Quantentheorie in Hydrodynamischer Form. *Z. Phys.*, 1926, **40**, 3/4, 322–328.
20. MADELUNG, E., *Die mathematischen Hilfsmittel des Physikers B*. Springer, Berlin, 1953.
21. BALESCU, R., Strange diffusion. *Cond. Matter Phys*. 1988, **1**, 4, 815–833.
22. ARKADIEV, V. A., POGREBKOV, A. K. and POLIVANOV, M. C., Inverse scattering transform method and soliton solutions for Davey–Stewartson II equation. *Physica. Ser. D.* 1989, **36**, 1–2, 189–197.

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